

On Genus of Circulant Graphs*

J. E. Strapasson[†]S. I. R. Costa[‡]M. M. S. Alves[§]

April 5, 2010

Abstract

Properties of circulant graphs have been studied by many authors, but just a few results concerning their genus characterization were presented up to now. We can quote the classification of all circulant planar graphs given by C. Heuberger in 2003. We present here a complete classification of circulant graphs of genus one, derive a general lower bound for the genus of a circulant graph and construct a family of circulant graphs which reach this bound.

Keywords. circulant graph, genus of a graph, flat torus, tessellation

AMS subject classifications. 05C50, 05E20, 52C07, 05C10

The genus of a graph, defined as the minimum genus of a 2-dimensional surface on which this graph can be embedded without crossings ([1, 2]), is well known as being an important measure of the graph complexity and it is related to other invariants like the algebraic connectivity of a graph ([3, 4, 5]).

A circulant graph, $C_n(a_1, \dots, a_k)$, is an homogeneous graph which can be represented (with crossings) by n vertices on a circle, with two vertices being connected if only if there is jump of a_i vertices from one to the other (Figure 1). Different aspects of circulant graphs have been studied lately, either theoretically or through their applications in telecommunication networks and distributed computation [6, 7, 8, 9].

Concerning specifically to the genus of circulant graphs few results are known up to now. We quote [10] for a small class of toroidal (genus one) circulant graphs, [9], which establish a complete classification of planar circulant graphs, and the cases where the circulant graph is either complete or a bipartite complete graph ([11, 12, 13, 14]).

In [15] we show how any circulant graph can be viewed as a quotient of lattices and obtain as consequences that: i) for $k = 2$, any circulant graph must be either genus one or zero (planar graph) and ii) for $k = 3$, there are circulant graphs of arbitrarily high genus.

We present here a complete classification of circulant graphs of genus one (Proposition 5), derive a general lower bound for the genus of a circulant graph $C_n(a_1, \dots, a_k)$ as $\frac{(k-2)n+4}{4}$, (Proposition 6), and construct a family of circulant graphs which reach this bound (Proposition 8).

This paper is organized as follows. In Section 2 we introduce concepts and previous results concerning circulant graphs and genus. In Section 3 we establish the classification of all circulant graphs of genus one (Propositions 4 and 5). In Section 4 we derive a lower bound for the genus of an n -circulant graphs of order $2k$ (Proposition 6) and construct families of graphs reaching this bound first for $k = 3$ (Proposition 7) and then for arbitrarily k (Proposition 8).

*This work was partially supported by Brazilian foundations FAPESP and CNPq.

[†]FAPESP 2007/00514-3 - Faculty of Applied Sciences, University of Campinas, Brazil (joao.strapasson@fca.unicamp.br).

[‡]CNPq 312061/2006-4 and FAPESP 2007/56052-8 - Institute of Mathematics, University of Campinas, Brazil (sueli@ime.unicamp.br)

[§]Department of Mathematics, Federal University of Paraná, Brazil (marcelomsa@ufpr.br)

1 Notation and Previous Results

In this section we summarize the notations and some concepts and results used in this paper concerning circulant graphs.

A *circulant graph* $C_n(a_1, \dots, a_k)$ with n vertices v_0, \dots, v_{n-1} and jumps a_1, \dots, a_k , $0 < a_j \leq \lfloor n/2 \rfloor$, $a_i \neq a_j$, is an undirected graph such that each vertex v_j , $0 \leq j \leq n-1$, is adjacent to all the vertices $v_{j \pm a_i \bmod n}$, for $1 \leq i \leq k$. A circulant graph is homogeneous: any vertex has the same order (number of incident edges), with is $2k$ except when $a_j = \frac{n}{2}$ for some j , when the order is $2k-1$.

The n -cyclic graph and the complete graph of n vertices are examples of circulant graphs denoted by $C_n(1)$ and $C_n(1, \dots, \lfloor n/2 \rfloor)$, respectively. Figure 1 shows on the left the standard picture of the circulant graph $C_{13}(1, 6)$.

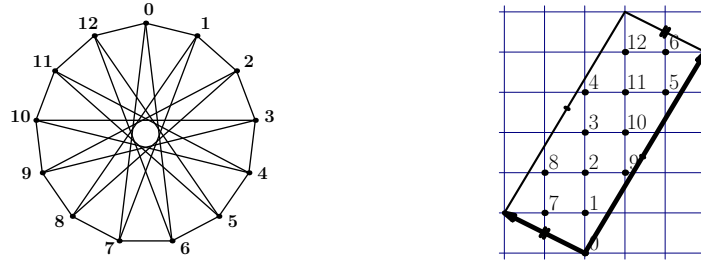


Figure 1: The circulant graph $C_{13}(1, 6)$ represented in the standard form (left) and on a 2-dimensional flat torus (right).

Two graphs are called isomorphic if there is a bijective mapping between their set of vertices which preserves adjacency. Two isomorphic graphs will be identified. In what follows we write $(a_1, \dots, a_k) = (\tilde{a}_1, \dots, \tilde{a}_k) \bmod n$ to indicate that for each i , there is j such that $a_i = \pm \tilde{a}_j \bmod n$. Two circulant graphs, $C_n(a_1, \dots, a_k)$ and $C_n(\tilde{a}_1, \dots, \tilde{a}_k)$ will be said to satisfy the *Ádám's relation* if there is r , with $\gcd(r, n) = 1$, such that

$$(a_1, \dots, a_k) = r(\tilde{a}_1, \dots, \tilde{a}_k) \bmod n \quad (1)$$

An important result concerning circulant graphs isomorphisms is that circulant graphs satisfying the Ádám condition are isomorphic ([16]). The reciprocal of this statement was also conjectured by Ádám. It is false for general circulant graphs but it is true in special cases such as $k = 2$ or $n = p$ (prime) (see [6]).

Without loss of generality we will always consider $a_1 < \dots < a_k \leq n/2$ for a circulant graph $C_n(a_1, \dots, a_k)$.

A circulant graph $C_n(a_1, \dots, a_k)$ is connected if, and only if, $\gcd(a_1, \dots, a_k, n) = 1$ ([10]). *In this paper we just consider connected circulant graphs.*

The genus of a graph is defined as the minimum genus, g , of a 2-dimensional orientable compact surface M_g on which this graph can be embedded without crossings ([1, 2]). This number, besides being a measure of the graph complexity, is related to other invariants like the algebraic connectivity ([3]).

A genus zero graph is also called a planar graph since through the stereographic projection there is also an embedding of this graph in the Euclidean plane.

A graph H is a *subgraph* of Γ if all its vertices and edges are also vertices and edges of Γ . E is an *expansion* of H if it is constructed from H by possibly adding new vertices on the edges of H . Finally, if

there is an expansion E of H which is a subgraph of G we say G is *supergraph* of H . From this definition follows that if G is a supergraph of H , $\text{genus}(G) \geq \text{genus}(H)$.

When a connected graph G is embedded on a surface, \mathcal{M}_g , of minimum genus g it splits the surface in regions called *faces*, each one homeomorphic to an open disc surrounded by the graph edges, giving rise to a tessellation on this surface. Denoting the number of faces, edges and vertices by f , e , and v respectively, those numbers must satisfy the well known Euler's second relation:

$$v + f - e = 2 - 2g \quad (2)$$

We quote next other known relations those numbers must satisfy ([1, 2]):

If G is a graph of genus g with $v \geq l$ such that any face in \mathcal{M}_g has at least l sides in its boundary,

$$lf \leq 2e \quad (3)$$

and

$$g \geq \frac{l-2}{2l}e - \frac{1}{2}(v-2). \quad (4)$$

In the above expressions we have equalities if, only if, all the faces have l sides.

An upper bound for the genus of a connected graph of n vertices is given by the genus of the *complete graph*, $C_n(1, \dots, \lfloor n/2 \rfloor)$, which is $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$. Combining the lower bound above with a minimum of three edges for each face, we can write the following inequality, for $n \geq 3$:

$$\left\lceil \frac{1}{6}e - \frac{1}{2}(n-2) \right\rceil \leq g \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \quad (5)$$

where $\lceil x \rceil$ is the ceiling (smallest integer which is greater or equal to) of x .

For a circulant graph $C_n(a_1, \dots, a_k)$, $a_1 < a_2 < \dots < a_k$ we can replace e by $e = nk$ when $a_k < \frac{n}{2}$, or $e = n(2k-1)/2$ when $a_k = \frac{n}{2}$. We can then rewrite the lower bound in last expression as $\left\lceil \frac{n}{6}(k-3) + 1 \right\rceil$ or $\left\lceil \frac{n}{6}(k-4) + 1 \right\rceil$, respectively.

1.1 Previous results on genus of circulant graphs

- The genus of the *complete bipartite graph* $K_{m,n}$ is known to be $\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$. Since $K_{n,n}$ is the circulant graph $C_{2n}(1, 3, \dots, 2\left\lceil \frac{n-1}{2} \right\rceil - 1)$, the genus of this one-parameter family is

$$\left\lceil \frac{(n-2)^2}{4} \right\rceil.$$

- *Planar circulant graphs* ($g = 0$). C. Heuberger gives in [9] a complete classification of planar circulant graphs

Proposition 1 ([9]) *A planar circulant graph is either the graph $C_n(1)$, or $C_n(a_1, a_2)$, where i) $a_2 = \pm 2a_1 \pmod n$ and $2|n$, ii) $a_2 = n/2$, and $2|a_2$.*

Figures 2 and 3 illustrate the planar circulant graphs of order 3 and 4, respectively.

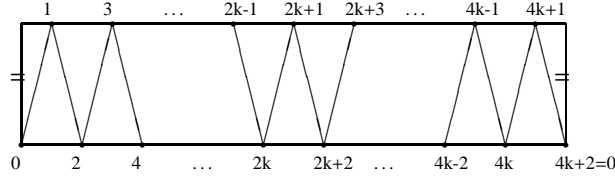


Figure 2: $C_n(1, 2)$, an anti-prism planar graph, viewed on cylinder.

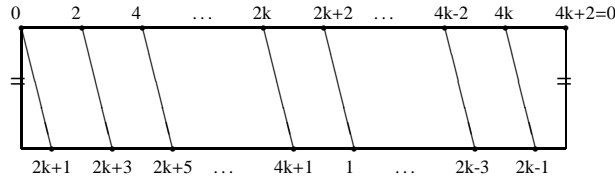


Figure 3: The circulant graph $C_{4k+2}(2, 2k+1)$, viewed on cylinder.

- For $k = 2$, and general (a_1, a_2) , we have shown that circulant graphs $C_n(a_1, a_2)$ are very far from reaching the upper bound for the genus given in (5), as was shown in [15]:

Proposition 2 ([15]) Any circulant graph $C_n(a_1, a_2)$, $a_1 < a_2 \leq n/2$, has genus one, except for the cases of planar graphs: i) $a_2 = \pm 2a_1 \pmod n$, and $2|n$, ii) $a_2 = n/2$, and $2|a_2$.

- For $k = 3$ and $n \neq 2a_3$ we can assert that the genus of $C_n(a_1, a_2, a_3)$ satisfies:

$$1 \leq g \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad (6)$$

The genus of the complete graph $C_7(1, 2, 3)$ achieves the minimum value one (5). However, in opposition to the case $k = 2$, the genus of a circulant graph $C_n(a_1, a_2, a_3)$ can be arbitrarily high:

Proposition 3 ([15]) There are circulant graphs $C_n(a_1, a_2, a_3)$ of arbitrarily high genus. A family of such graphs is given by: $n = (2m+1)(2m+2)(2m+3)$, $m \geq 2$; $a_1 = (2m+2)(2m+3)$, $a_2 = (2m+1)(2m+2)(m+1)$, $a_3 = (2m+2)(2m+3)(m+1)$, with the correspondent genus satisfying

$$g \geq 2m(m+1)^2 + 1. \quad (7)$$

In the next section we classify all circulant graphs of genus one and in Section 4 we deduce a general lower bound for the genus of circulant graphs and construct a family of graphs which reach this bound.

2 Toroidal Circulant Graphs

This section is devoted to classify all toroidal (genus one) circulant graphs. The next proposition describes all circulant graphs of degree 5 or 6 of genus one.

Proposition 4 *The genus of a circulant graph $C_n(a_1, a_2, a_3)$ of degree 5 or 6, $0 < a_1 < a_2 < a_3 \leq \frac{n}{2}$ satisfies:*

- 1) $\text{genus}(C_n(a_1, a_2, a_3)) \geq 1$, $\forall a_1, a_2, a_3$;
- 2) $\text{genus}(C_n(a_1, a_2, a_3)) = 1$ if, and only if,
 - i) $a_3 = a_1 + a_2$, or
 - ii) $a_2 = 2a_1$, $n = 2a_3$ for a_1 and a_3 odd numbers, or
 - iii) $C_8(a_1, a_2, a_3) \equiv C_8(1, 2, 4)$.

Proof. Item 1 was proved in [9] as mentioned in the last section. We show next that the genus of all circulant graphs satisfying the conditions of item 2 is one.

(i) We can assert $C_n(a_1, a_2, a_1 + a_2)$ has genus one, since it is isomorphic to the supergraph of $C_n(a_1, a_2)$ where the vertices are preserved and the new added edges correspond to the main diagonals of the squares which tessellate the flat torus defined by $C_n(a_1, a_2)$ ([15], Proposition 3). Hence $C_n(a_1, a_2, a_1 + a_2)$ will tessellate the same flat torus by including subdivisions into triangles.

(ii) Suppose that a_1 and a_3 are odd, $a_2 = 2a_1$ and $n = 2a_3$. From $\gcd(a_1, a_2, a_3, n) = 1$ we get $\gcd(a_1, 2a_1, \frac{n}{2}, n) = 1$ and hence $\gcd(a_1, n) = 1$. On the other hand, since $\frac{n}{2} = b \frac{n}{2} \pmod{n}$, for any b such that $\gcd(b, n) = 1$, by Ádám relation we have $C_n(a_1, a_2, a_3) \equiv C_n(1, 2, a_3)$. The circulant graph $C_n(1, 2, a_3)$ is a supergraph of the antiprism graph $C_n(1, 2)$. The new edges to be added can be placed in a tube connecting the two $n/2$ -polygons which are the antiprism floor and ceiling. Figure 4 illustrates the graph $C_n(1, 2, a_3)$ on a flat torus flattened.

(iii) $C_8(1, 2, 4)$ is a supergraph of the planar graph $C_8(1, 2)$ which can be viewed as an antiprism placed on a cylinder. This is shown in Figure 5. The upper square vertices of this antiprism are labelled by even numbers, whereas the vertices on the bottom square are labelled by odd numbers. As we can see, the circulant graph new edges corresponding to $a_3 = 4$ can be added on the torus in the way showed in this figure.

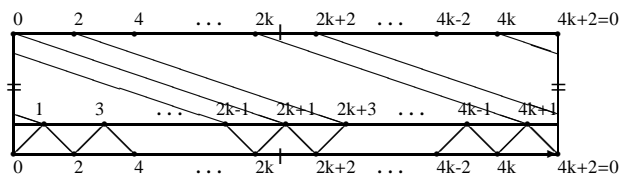
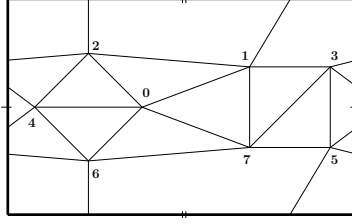


Figure 4: Embedding of the circulant graph $C_{4k+2}(1, 2, 2k+1)$ on a flat torus.

Figure 5: Embedding of the circulant graph $C_8(1, 2, 4)$ on a flat torus.

To see that no other circulant graph $C_n(a_1, a_2, a_3)$ can be embedded on a torus we consider two cases:

a) $C_n(a_1, a_2, a_3)$ has degree 6 ($a_3 \neq n/2$). If $C_n(a_1, a_2, a_3)$ has genus one, either $C_n(a_1, a_2)$ or $C_n(a_1, a_3)$ must also have genus one, since the planarity conditions given in Proposition 1 cannot be fulfilled ($a_3 \neq \frac{n}{2}$ and either $a_2 \neq 2a_1$ or $a_3 \neq 2a_1$). Hence, either $C_n(a_1, a_2)$ or $C_n(a_1, a_3)$ must tessellate the torus and, by Proposition 3 of [15], this tessellation can be done by squares. The only possibility for $C_n(a_1, a_2, a_3)$ is to be a supergraph obtained from either $C_n(a_1, a_2)$ adding an edge on the main diagonals of the squares ($a_3 = a_1 + a_2$) or from $C_n(a_1, a_3)$ adding an edge on the skew diagonals of the squares ($a_2 = a_3 - a_1$).

b) $C_n(a_1, a_2, a_3)$ has degree 5 ($a_3 = n/2$). If $C_n(a_1, a_2)$ has genus one, it tessellates a torus by squares, the proof is as in a). If $C_n(a_1, a_2)$ is planar, by Proposition 1 we must have either $\gcd(a_1, n) = 1$ or a_1 even. In the first case, $C_n(a_1, 2a_1, \frac{n}{2}) \equiv C_n(1, 2, \frac{n}{2})$, since $(a_1, 2a_1, \frac{n}{2}) = a_1(1, 2, \frac{n}{2}) \pmod{n}$. If $\frac{n}{2}$ is odd, the condition 2-ii) is verified. If $\frac{n}{2}$ is even, then $n = 4u$, $C_n(a_2, a_3) = C_{4u}(2, 2u)$ which has two connected components both isomorphic to $C_{2u}(1, u)$ and those components are non-planar except for $u = 2$ ([9]). We know that for $u = 2$ the genus is one, since condition 2-iii) is satisfied. On the other hand if $u > 2$ we can assert that $C_{4u}(1, 2, 2u)$ must have genus at least two since it is a supergraph of two disconnected subgraphs of genus one. In the second case, a_1 even, we have $\gcd(a_1, n) = 2$ and then $\gcd(a_1/2, n) = 1$ which implies $C_n(a_1, 2a_1, n/2) \equiv C_n(2, 4, n/2)$ where $n/2 = 2u + 1$. Hence $C_n(2, 4)$ is a disconnected graph where each component is isomorphic to $C_{n/2}(1, 2)$, which is a genus one graph (Proposition 2). Therefore we conclude that $C_n(2, 4)$ has genus at least two and then the same holds for its supergraph $C_n(2, 4, n/2)$. This completes the proof. \square

In the next proposition we classify all genus one circulant graphs:

Proposition 5 (Classification of genus one circulant graphs) *The circulant graphs of genus one are, up to isomorphism:*

- 1) $C_n(a_1, a_2)$, $a_1 < a_2 \leq \frac{n}{2}$, where either
 - i) n is odd, or
 - ii) n is even and a_2 is odd or
 - iii) n and a_2 are even but $n \neq 2a_2$ and $a_2 \neq 2a_1$.
- 2) $C_n(a_1, a_2, a_3)$, $a_1 < a_2 < a_3 \leq \frac{n}{2}$, and
 - i) $a_3 = a_1 + a_2$, or
 - ii) $a_2 = 2a_1$, $n = 2a_3$ for a_1 and a_3 odd, or
 - iii) $C_8(a_1, a_2, a_3) \equiv C_8(1, 2, 4)$.

Proof. The proof is almost done. $C_n(1)$ is planar. For $k = 2$, condition 1) is just a translation of conditions of Proposition 2 written in the negative form. Condition 2) was the required in the last proposition for $k = 3$. So, it remains to prove that no other circulant graph has genus one. In fact, if $k \geq 4$ we must have from (4):

$$g \geq \left\lceil \frac{e}{6} - \frac{v-2}{2} \right\rceil \geq \left\lceil \frac{(2k-1)n}{12} - \frac{n-2}{2} \right\rceil \geq \left\lceil \frac{7n}{12} - \frac{n-2}{2} \right\rceil \geq \left\lceil \frac{n}{12} + 1 \right\rceil \geq 2.$$

□

Remark 1 The algebraic connectivity $a(G)$, i.e., the second smallest eigenvalue of the Laplacian matrix, is a spectral property of a graph G , which is an important parameter in the analysis of various robustness-related problems [5]. In [4] it is derived an upper bound for the algebraic connectivity related to the graph genus: $a(G) \leq \frac{(6g+2)\Delta(G)}{\sqrt{n/2-3(g+2)}}$ for $n \geq 18(g+2)^2$, where $\Delta(G)$ is the maximum vertex degree of G . For a circulant graph $C_n(a_1, a_2, a_1 + a_2)$ this upper bound is $\frac{96}{-18 + \sqrt{2}\sqrt{n}}$. It is interesting to check the tightness of this bound for the above family $C_{2^j}(1, \lfloor \sqrt{2^j} \rfloor - 1, \lfloor \sqrt{2^j} \rfloor)$. The relevance of this bound is, of course, for big values of n when it is not viable to explicitly calculate neither the algebraic connectivity nor the diameter $D(G)$ of those graphs.

n	$\lfloor n \rfloor - 1$	$D(G)$	$\frac{4}{nD(G)}$	$a(G)$	$\frac{96}{\sqrt{2}\sqrt{n} - 18}$
256	15	16	0.000976563	0.286858	20.7459
512	21	19	0.000411184	0.134179	6.85714
1024	31	32	0.00012207	0.0745394	3.52231
2048	44	35	0.0000558036	0.0298565	2.08696
4096	63	64	0.0000152588	0.0189651	1.32396
8192	89	89	$5.48631 \cdot 10^{-6}$	0.00942163	0.872727
16384	127	na	na	na	0.588887
32768	180	na	na	na	0.403361
65536	255	na	na	na	0.279038
131072	361	na	na	na	0.194332
262144	511	na	na	na	0.135962
524288	723	na	na	na	0.0954274

3 A lower bound for the genus of circulant graphs and a family of graphs which reach this bound.

In this section we consider circulant graphs $C_n(a_1, \dots, a_k)$, $k \geq 3$, establish a lower bound for the genus of those graphs and construct a family of circulant graphs which reach this lower bound.

We remark that, from Euler relation, concerning circulant graph embeddings the greater the number of faces determined the smaller the genus, roughly speaking. Hence looking for circulant graphs which generate mostly triangular and quadrilateral faces in their embeddings we have more chances of obtaining smaller genus.

A condition required for the existence of a triangular face in a $C_n(a_1, \dots, a_k)$ embedding, $k \geq 2$, is $a_l = a_i + a_j$ for some i, j, l , since we must have a three-step loop on the correspondent vertices. Note that this condition appears in the classification of planar and toroidal graphs.

Considering the limit case where the above condition is not satisfied and hence all the faces induced by the circulant graph embedding have at least four sides we may write, using (4):

$$g \geq \frac{l-2}{2l} a - \frac{v-2}{2} = \frac{2}{8} nk - \frac{n-2}{2} = \frac{nk - 2n + 4}{4}.$$

Hence, we get the following proposition, which establishes a lower bound for circulant graphs with no triangular faces.

Proposition 6 *The genus, g of the circulant graph $C_n(a_1, \dots, a_k)$, such that $a_i \neq a_j + a_l$, $\forall i, j, l \leq k$ and $n \neq 2a_i$, $\forall i$ satisfy:*

$$g \geq \frac{nk - 2n + 4}{4}.$$

Since under the conditions of the above proposition the lower bound can only be reached if all faces are “squared”, we present next a construction of a family of those graphs for $k = 3$ which will be then extended for any k in Proposition 8. In what follows the (additive) subgroup of \mathbb{Z}_n generated by $a_1, \dots, a_l \in \mathbb{Z}_n$ is denoted by $\langle a_1, \dots, a_l \rangle$. Let $0 < a_1 < \dots < a_k < n$, $d = \gcd(n, a_1, a_2, \dots, a_k)$, and let $G_s = \langle a_1, a_2, \dots, a_{s-1} \rangle \subset \mathbb{Z}_n$. Define L_s for $s = 2, \dots, k$ by $L_1 = o(a_1)$ is the order of a_1 in \mathbb{Z}_n and $L_s = \min\{t \in \mathbb{N}; t a_s \in G_s\}$ if $1 < s \leq k$.

Lemma 1 *Given $x \in \mathbb{Z}_n$, let r be the remainder of x in the division by d ; then there exist unique $m_i \in \mathbb{N}$ such that*

$$x = r + m_1 a_1 + \dots + m_s a_s$$

with $0 \leq m_i < L_s$ for all s .

Proof. Dividing x by d we get $x = qd + r$. Since $d = \gcd(n, a_1, a_2, \dots, a_k)$, there are $t_{1,1}, t_{1,2}, \dots, t_{1,s}$ such that

$$x = r + t_{1,1} a_1 + \dots + t_{1,s} a_s$$

in \mathbb{Z}_n . Dividing $t_{1,s}$ by L_s we get $t_{1,s} = q_s L_s + m_s$, we have $t_{1,s} a_s = q_s L_s a_s + m_s a_s$, where $q_s L_s a_s$ is in G_s ; therefore there are $t_{2,1}, \dots, t_{2,s-1} \in \mathbb{N}$ such that

$$x = r + t_{2,1} a_1 + \dots + t_{2,s-1} a_{s-1} + m_s a_s$$

with $0 \leq m_s < L_s$. Proceeding in this fashion, we obtain m_1, m_2, \dots, m_s such that

$$x = r + m_1 a_1 + \dots + m_s a_s$$

with $0 \leq m_i < L_i$ for each $i = 1, 2, \dots, s$.

This expression is unique: suppose that

$$r + m_1 a_1 + \dots + m_s a_s = r + t_1 a_1 + \dots + t_s a_s$$

with $0 \leq m_i < L_i$ and $0 \leq t_i < L_i$, and let j be the greatest index such that $m_i \neq t_i$; hence

$$r + m_1 a_1 + \dots + m_j a_j = r + t_1 a_1 + \dots + t_j a_j$$

and $m_j \neq t_j$, say, $m_j > t_j$. It follows that $(m_j - t_j)a_j \in G_j$, but $0 < m_j - t_j < L_j$, contradiction. Therefore $m_i = t_i$ for each i . \square

We will construct the embedding of $C_n(a_1, \dots, a_k)$ by induction on k . Note that $C_n(a_1, \dots, a_{k-1})$ may be disconnected: it is well known that this graph has d components, where $d = \gcd(n, a_1, \dots, a_{k-1})$, and that each component is isomorphic to $C_{\frac{n}{d}}(\frac{a_1}{d}, \dots, \frac{a_{k-1}}{d})$. Since x and y are linked by a path if and only if

$x - y = m_1 a_1 + \cdots + m_{k-1} a_{k-1}$ in \mathbb{Z}_n , it follows that x and y are in the same component if and only if $x \equiv y \pmod{d}$. Hence, each $0 \leq r < d$ determines a component and the numbers

$$r + m_1 a_1 + \cdots + m_{k-1} a_{k-1},$$

with $0 \leq m_i < L_i$, describe all the vertices of the component of $C_n(a_1, \dots, a_{k-1})$ associated to r .

Proposition 7 Let $G = C_n(a_1, a_2, a_3)$, where $n = 2^r l$, $a_1 = 2^{r_1} l_1$, $a_2 = 2^{r_2} l_2$, where l, l_1, l_2, a_3 odd, $0 < r_2 < r_1 < l$ and $a_i \neq \pm 2 a_{i+1}$, $i = 1, 2$. Hence, the genus of G is $\frac{n+4}{4}$.

Proof. Assume that $a_i \neq \pm 2 a_{i+1}$, $i = 1, 2$. As we have remarked previously, the graph G has no triangles and therefore its genus is at least $\frac{n+4}{4}$. We will prove that the genus is exactly $\frac{n+4}{4}$ by showing that this graph can be embedded on a surface of this same genus. Let us proceed to the construction of this embedding. We remind that all faces should have 4 edges.

Let $d = \gcd(n, a_1, a_2)$. The circulant graph $G = C_n(a_1, a_2)$ has d components H_r , $0 \leq r < d$, each isomorphic to $C_{\frac{n}{d}}(a_1/d, a_2/d)$. We have already seen that each component is a quadrilateral tessellation of a torus, i.e., all faces are bounded by 4-edges (Proposition 3 in [15]). We denote by T_r the torus that contains the vertex r . In order to place the edges corresponding to $\pm a_3$, we will connect these tori by “tubes”.

Note that the components must be embedded properly. If the embedding is constructed following figure 6, there will be necessarily faces with 6 edges (and also must be crossings of the edges on the tubes).

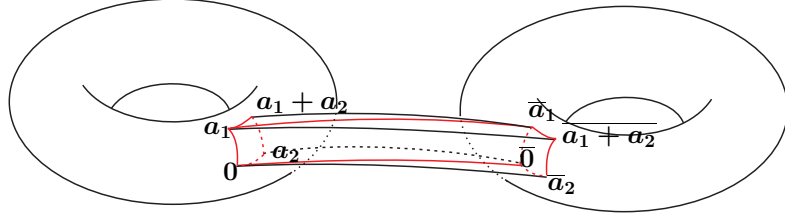


Figure 6: Embedding of a degree 6 circulant graph: The improper connection between the tori T_0 and T_{a_3} (\bar{x} denotes the vertex $x + a_3$).

However, if we reverse the orientation of the torus $T_{r \pm a_3}$, the adjoined faces have 4 edges, as it can be seen in figure 7.

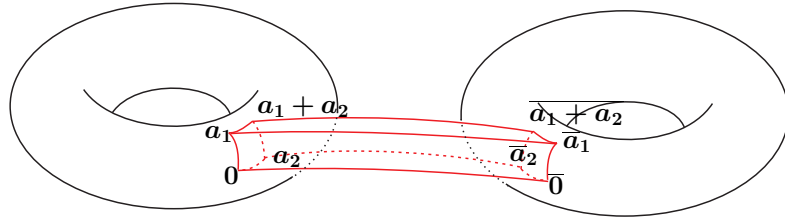


Figure 7: Embedding of a degree 6 circulant graph: The proper connection between the tori T_0 and T_{a_3} (\bar{x} denotes the vertex $x + a_3$).

Hence, we reverse the orientation of every component that contains an odd multiple of a_3 (these are the tori T_r where r is odd). This can be done because d is even.

Now we add the tubes, which can be seen as “squared”-base prisms. We choose as the lower base of each prism a face of T_r with r even, i.e., the vertices of the lower base have even labels of the form $P, P + a_1, P + a_1 + a_2, P + a_2$ and, therefore, the vertices of the upper base have odd labels of the form $I, I + a_1, I + a_1 + a_2, I + a_2$, with $I = P \pm a_3$. Since we are dealing with these tori as quotients of lattices, we must be careful when choosing the faces that are to be the bases of the prisms.

Given an even vertex x of the graph, we have

$$x = r + m_1 a_1 + m_2 a_2$$

where r is even, $0 \leq m_2 < L_2$ and $0 \leq m_1 < L_1 = o(a_1)$. Dividing m_i by 2 we get

$$x = r + (2Q_1 + \delta_1) a_1 + (2Q_2 + \delta_2) a_2$$

with $\delta_i = 0$ or 1. Therefore the vertex set of this component is the disjoint union of the squares $\{P, P + a_1, P + a_1 + a_2, P + a_2\}$, where P has the form

$$P = r + 2Q_1 a_1 + 2Q_2 a_2, \tag{8}$$

with r even, $0 \leq 2Q_1 < o(a_1) = L_1$ and $0 \leq 2Q_2 < L_2$ (L_i as in Lemma 1).

Now we connect the square $\{P, P + a_1, P + a_1 + a_2, P + a_2\}$ in T_r to the square $\{(P + a_3), (P + a_3) + a_1, (P + a_3) + a_1 + a_2, (P + a_3) + a_2\}$ in T_{r+a_3} by a prism whose edges, besides those of the bases, are the four edges of the form $[x, x + a_3]$, where x is a vertex of the base in T_r . Then we cut out the two bases. On this surface we embed all edges of the form $[v, v + a_3]$.

In the same manner, we can write each vertex x of H_r as

$$x = r + (2Q_1 + \sigma_1) a_1 + (2Q_2 + \sigma_2) a_2$$

with $\sigma_i = 0$ or -1 . Then the squares $\{P, P - a_1, P - a_1 - a_2, P - a_2\}$, where P is as in (8), also contain the vertex set of H_r . The faces of these squares are all distinct from the others already cut out from T_r . Hence, we can connect $\{P, P - a_1, P - a_1 - a_2, P - a_2\}$ in T_r to $\{P - a_3, P - a_3 - a_1, P - a_3 - a_1 - a_2, P - a_3 - a_2\}$ in T_{r-a_3} by a prism; the remaining edges are realized on the surface thus obtained. The genus of this surface is then obtained using Lemma 5, since every face of this tessellation is a quadrilateral, thus completing the proof. \square

Example 1 The graph $C_{32}(8, 2, 3)$ can be embedded on a 9-torus (a sphere with 9 handles), generating a tessellation by squares, as it is shown in Figure 8.

We now to generalize this result for $k > 3$. When $k = 3$ we have started from the embedding of squares of the subgraph $C_n(a_1, a_2)$ by squares. It is reasonable to expect that this works also for $k > 3$.

Proposition 8 Let $G = C_n(a_1, \dots, a_k)$, where $n = 2^r l$, $a_i = 2^{r_i} l_i$, $i = 1, \dots, k-1$, where l, l_i, a_k odd, $0 < r_{i+1} < r_i < l$ and $a_i \neq \pm 2 a_{i+1}$, $i = 1, \dots, k-1$. Hence, the genus of G is $\frac{nk - 2n + 4}{4}$.

Proof. The proof will be done by induction on k . For $k = 3$, it is Proposition 7 above. Assume that the result holds for $k-1$. The graph $H = C_n(a_1, \dots, a_{k-1})$ is a disconnected circulant graph with an even number of connected components, H_r , $r = 0, 1, \dots, d-1$, where $d = \gcd(n, a_1, \dots, a_{k-1})$. Each component H_r can be

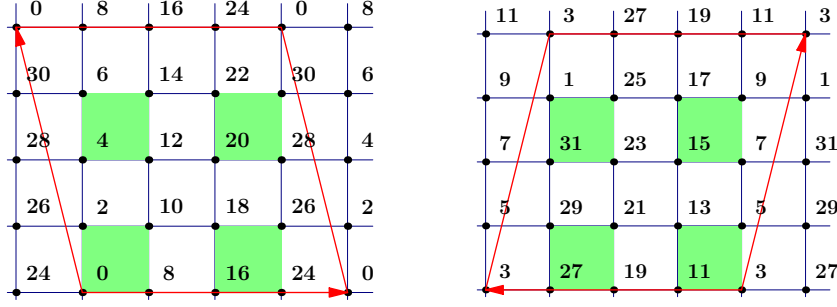


Figure 8: This figure shows the two connected components of $C_{32}(8, 2)$. Using the embedding given by Proposition 3 of [15], we reverse the orientation of the second component.

embedded on a surface S_r giving rise tessellation where every face has 4 edges. As in the last proposition, we reverse the orientation of the components that contain the odd multiples of a_k . We wish to add tubes that, topologically, are prisms with squared bases.

Let $x \in H_r$, r even; then $x = r + m_1 a_1 + \dots + m_{k-1} a_{k-1}$ where $0 \leq r < d$ and $0 \leq m_i < l_i$ as in Lemma 1. Given $0 < j < k - 1$, we can write

$$x = r + (2p_j + \delta_j) a_j + (2p_{k-1} + \delta_{k-1}) a_{k-1} + \sum_{\substack{i=1, \dots, k-2 \\ i \neq j}} p_i a_i \quad (9)$$

with $\delta_j, \delta_{k-1} \in \{0, 1\}$. Then each vertex of H_r is a vertex of a square determined by $\{P, P + a_{k-1}, P + a_{k-1} + a_j, P + a_j\}$, where P is of the form

$$P = r + 2p_j a_j + 2p_{k-1} a_{k-1} + \sum_{\substack{i=1, \dots, k-2 \\ i \neq j}} p_i a_i \quad (10)$$

where $2p_j < l_j$, $2p_{k-1} < l_{k-1}$ and $p_s < l_s$, $s \neq j, k - 1$. Just as in Proposition 7, each such square is then connected to the square $\{P + a_k, P + a_k + a_{k-1}, P + a_k + a_{k-1} + a_j, P + a_k + a_j\}$ (which lies in H_{r+a_k}) by a prism which contains the edges $[P, P + a_k]$, $[P + a_{k-1}, P + a_k + a_{k-1}]$, $[P + a_{k-1} + a_j, P + a_k + a_{k-1} + a_j]$ and $[P + a_j, P + a_k + a_j]$; and then we cut out both squares. Doing this for every $j \in \{1, 2, \dots, k - 2\}$ we construct a surface where each edge of the form $[x, x + a_k]$ is embedding without crossings.

In order to add the edges of the form $[x, x - a_k]$ we work with the squares $\{P, P - a_{k-1}, P - a_{k-1} - a_j, P - a_j\}$ and $\{P - a_k, P - a_k - a_{k-1}, P - a_k - a_{k-1} - a_j, P - a_k - a_j\}$. By the same reasoning, we can connect these squares by prisms and construct a surface that is tessellated by $C_n(a_1, \dots, a_k)$, where each face is a square. This last remark shows that this surface has the required genus, and this concludes the proof. \square

The next example shows that there are more circulant graphs than the ones considered in Proposition 8 which also can be embedded giving rise to a quadrilateral tessellation.

Example 2 For the graph $C_{32}(8, 2, 3, 7)$, if we consider $C_{32}(8, 2, 3)$ as in last proposition, we note that just half the faces of the tessellation of $C_{32}(8, 2)$ are excluded to add tubes. We can also exclude the other faces adding tubes to support the edges $\pm a_4$. Hence this is an embedding generating quadrilateral faces and since there are no cycles of size 3, the expression $\frac{nk - 2n + 4}{4}$ for the genus still holds.

Figure 9 shows all the circulant graphs of 32 vertices for which the genus can be given by Propositions 1 (Heuberger), 5 and 8.

k	$a_1 \in$	$a_2 \in$	$a_3 \in$	$a_4 \in$	g
1	I	—	—	—	0
2	I	$2(I - \{\pm a_1\})$	—	—	1
2	I	$4I$	—	—	1
2	I	$8I$	—	—	1
3	I	$2(I - \{\pm a_1\})$	$4(I - \{\pm 2a_2\})$	—	9
3	I	$2(I - \{\pm a_1\})$	$8I$	—	9
3	I	$4I$	$8(I - \{\pm 2a_2\})$	—	9
4	I	$2(I - \{\pm a_1\})$	$4(I - \{\pm 2a_2\})$	$8(I - \{\pm 2a_3\})$	17

Figure 9: All circulant graphs of 32 vertices satisfying 1 (Heuberger), 5 and 8 ($I = \{\pm 1, \pm 3, \dots, \pm 15\}$).

4 Acknowledgment

This work was partially supported by FAPESP-Brazil 2007/56052-8, Fapesp 2007/00514-3 and CNPq 312061/2006-4.

References

- [1] J. E. Gross and T. W. Tucker, Topological graph theory Dover Publications Inc., Mineola, NY, 2001.
- [2] R. J. Trudeau, *Introduction to graph theory*. Dover Publications Inc., New York, 1993. Corrected reprint of the 1976 original.
- [3] J. J. Molitierno, On the algebraic connectivity of graphs as function of genus, Linear Algebra Appl., 419 (2-3) (2006), 519-531.
- [4] A. G. Boshier, Enlarging properties of graphs, Ph.D. Thesis, Royal Holloway and. Bedford New College, University of London, 1987.
- [5] A. Jamakovic and S. Uhlig, On the relationship between the algebraic connectivity and graphs robustness to node and link failures. Next Generation Internet Networks, 3rd EuroNGI Conference on, Trondheim, Norway (2007)
- [6] V. Liskovets and R. Pöschel, Counting circulant graphs of prime-power order by decomposing into orbit enumeration problems. Discrete Math., 214(1-3) (2000), 173-191.
- [7] M. Muzychuk, Ádám's conjecture is true in the square-free case, J. Combin. Theory Ser. A 72 (1) (1995), 118-134.
- [8] F. P. Muga II, Undirected Circulant Graphs, International Symposium on Parallel Architectures, Algorithms and Networks, (1994), 113-118.
- [9] C. Heuberger, On planarity and colorability of circulant graphs. Discrete Math., 268(1-3) (2003), 153-169.
- [10] F. Boesch and R. Tindell, Circulants and their connectivities. J. Graph Theory, 8 (4) (1984), 487-499.

- [11] L. W. Beineke and F. Harary, The Genus of the n -Cube. *Canad. J. Math.* 17 (1965), 494-496.
- [12] F. Harary, *Graph Theory*. Reading, MA: Addison-Wesley, 1994.
- [13] G. Ringel, Das Geschlecht des vollständiger Paaren Graphen. *Abh. Math. Sem. Univ. Hamburg*, 28 (1965), 139-150.
- [14] G. Ringel and J. W. T. Youngs, Solution of the Heawood Map-Coloring Problem. *Proc. Nat. Acad. Sci. USA*, 60 (1968), 438-445.
- [15] S. I. R Costa, J. E Strapasson, M. Muniz, and T. B. Carlos, Circulant graphs and tessellations on flat tori. *Linear Algebra and Appl*, 432-1(2010), 369-382
- [16] A. Ádám, Research problem 2-10. *J. Combinatorial Theory*, 1967.